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On the number of spanning trees and Eulerian tours in iterated line digraphs ☆

Zhang Huaxiao [a], Zhang Fuji [b], Huang Qiongxiang [a,*]

[a] Department of Mathematics, Xinjiang University, Urumqi, Xinjiang 830046, People's Republic of China

[b] Department of Mathematics, Xiamen University, Xiamen 361005, People's Republic of China

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Abstract

This paper provides closed-form expressions for the number of directed spanning trees and Eulerian tours in iterated line graphs of regular digraphs. Some applications are also considered.

1. Introduction

In recent years, motivated by the design of local area networks, communication networks and other distributed computer systems, the theory of interconnected networks has fast developed [1, 2, 10]. These networks can be modeled by directed or undirected graphs. Fiol et al. [5] considered the line digraph interation which produces several types of interconnected networks such as the Kautz graphs, de Bruijn graphs and some other types of digraphs and they discussed the maximum distance between any pair of vertices and the average distance between vertices. A simple local routing algorithm was also provided. Two of the present authors determined the homomorphisms and automorphisms of the Kautz graphs and de Bruijn graphs (for details see [8]). Some other results on this subject can be found in [2, 9]. In this paper we show that the number of directed spanning trees and Eulerian tours of iterated line digraph of regular digraph D can be obtained by the number of directed spanning trees of D. Using these results we calculate the number of spanning trees of several types of interconnected networks. Three new results (Examples 2–4) and an old result are deduced.

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* Corresponding author.

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2. Main results

A digraph (directed graph) is defined to be a pair $(V(D), A(D))$, where $V(D)$ is a finite set of vertices and $A(D)$ is the set of arcs which is a family of ordered pairs of elements of $V(D)$. The arc (a, a) is called a self-loop (the concept of self-loops is useful in de Bruijn graphs). The line digraph $L(D)$ of digraph D has as its vertex set the family of arcs of D. For $e_1, e_2 \in A(D)$, (e_1, e_2) is an arc of $L(D)$ iff there are vertices a_1, a_2, a_3 in D with $e_1 = (a_1, a_2)$ and $e_2 = (a_2, a_3)$. In other words, the head of e_1 is the tail of e_2. It is clear that $L(D)$ is a digraph and $L(D)$ has a self-loop at vertex (a, a) iff (a, a) is a self-loop of D. The iterated line digraphs of D are defined recursively as follows: $L^k(D) = L(L^{k-1}(D)$, $k \geqslant 2$ (see Fig. 1).

Let $D = (V(D), A(D))$ be a digraph. For $i \in V(D)$ we define its outdegree $d^+(i) = |\{j : (i, j) \in A(D)\}|$ and indegree $d^-(i) = |\{j : (j, i) \in A(D)\}|$. D is said to be k-regular if $d^+(i) = d^-(i) = k$, $\forall i \in V(D)$. The adjacency matrix of digraph D is a matrix $A = (a_{ij})$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, & (i, j) \in A(D), \\ 0, & \text{otherwise.} \end{cases}$$

The directed tree matrix of D denoted by $M(D) = (m_{ij})$ is a matrix of order $|V(D)|$, where m_{ii} denotes the outdegree of vertex i and $m_{ij} = -a_{ij}$ $(i \neq j)$, after all the self-loops if any have been removed from D. The characteristic polynomial of D is defined to be

$$P(D, \lambda) = |\lambda I - A|.$$

Now we consider the number of directed spanning trees of $L^r(D)$. First we need the following known lemmas.

Lemma 1 ([Fuji and Guoning 7, Theorem 1]). *Let $D = (V(D), A(D))$ be a digraph with characteristic polynomial $P(D, \lambda)$ and $L(D)$ be its line digraph with characteristic polynomial $P(L(D), \lambda)$; then*

$$P(L(D), \lambda) = \lambda^{|A(D)| - |V(D)|} P(D, \lambda).$$

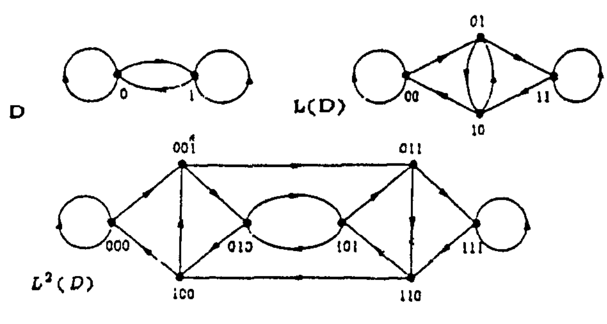

Fig. 1.

Lemma 2 (Chen[4, Corollary 4.4]). For a given digraph D the value of cofactor m_{ii} of its directed tree matrix $M(D)$ is equal to the number of directed spanning trees rooted at vertex i.

A square matrix is said to be equicofactor if it has the properties that the sum of the entries of every row and every column equals zero. One can see that if D is a k-regular digraph then its directed matrix $M(D)$ is equicofactor.

Lemma 3 (Chen[4, Theorem 4.1]). If $M(D)$ is a equicofactor matrix then all the cofactors of the entries of $M(D)$ are equal.

Lemma 4. *Let D be a k-regular digraph, then D has the same number of directed spanning trees rooted at each vertex.*

Proof. Note that the number of directed spanning trees rooted at vertex i is equal to the value of the cofactor m_{ii}'s in $M(D)$. By Lemma 3 the values of cofactor of m_{ii} are equal. The lemma is thus proved. □

The following Lemma is a special case of Theorem 5.4 in [4].

Lemma 5. *Let D be a connected k-regular digraph; then the number of its Eulerian tours is*

$$E(D) = K(D)[(k-1)!]^n.$$

where $K(D)$ is the number of spanning trees of D rooted at each vertex.

Now we are in the position to prove our main results.

Theorem 1. *Let D be a k-regular digraph with n vertices; then the number of directed spanning trees of $L^r(D)$ is*

$$T(L^r(D)) = k^{nk^r - n}T(D).$$

where $T(D)$ is the number of directed spanning trees of D.

Proof. By Lemma 1, the characteristic polynomial of line digraph of D is

$$P(L(D), \lambda) = \lambda^{kn-n}P(D, \lambda).$$

Since $L^r(D)$ has nk^r vertices and nk^{r+1} arcs, by induction on r, we have

$$P(L^r(D), \lambda) = \lambda^{nk^r - n}P(D, \lambda).$$

Since D is a k-regular digraph, we have $P(D, k) = 0$. Taking the derivative with respect to λ and envaluating at $\lambda = k$, we obtain

$$P'(L^r(D), k) = k^{nk^r - n}P'(D, k). \tag{1}$$

Since the characteristic polynomial of D is

$$P(D,\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix},$$

we have

$$P'(D,k) = \begin{vmatrix} k - a_{22} & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & k - a_{33} & \cdots & -a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n2} & a_{n3} & \cdots & k - a_{nn} \end{vmatrix}$$

$$+ \begin{vmatrix} k - a_{11} & -a_{13} & \cdots & -a_{1n} \\ -a_{31} & k - a_{33} & \cdots & -a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & a_{n3} & \cdots & k - a_{nn} \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} k - a_{11} & -a_{12} & \cdots & -a_{1\,n-1} \\ -a_{21} & k - a_{22} & \cdots & -a_{2\,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n-1\,1} & -a_{n-1\,2} & \cdots & k - a_{n-1\,n-1} \end{vmatrix}.$$

By the definition of $M(D)$ and Lemmas 2–4, we have

$$P'(D,k) = T(D). \tag{2}$$

Similarly,

$$P'(L^r(D),k) = T(L^r(D)).$$

Comparing it with (1) and (2), the theorem readily follows. □

For a regular digraph D, the following theorem gives the number of Eulerian tours in its iterated line digraph.

Theorem 2. *Let D be a k-regular digraph with n vertices. The number of Eulerian tours of $L^r(D)$ is*

$$E(L^r(D)) = \frac{(k!)^{nk^r}}{nk^{r+n}} T(D).$$

Proof. Since $L^r(D)$ is k-regular with nk^r vertices, by Lemma 4, D has the same number of directed spanning trees rooted at each vertex. Hence, by Theorem 1, the number of directed spanning trees rooted at a fixed vertex is

$$K(L^r(D)) = \frac{1}{nk^r} T(L^r(D)) = \frac{1}{n} k^{nk^r - r - n} T(D).$$

By Lemma 5, the number of Eulerian tours of $L^r(D)$ is

$$E(L^r(D)) = K(L^r(D))((k-1)!)^{nk^r} = \frac{1}{n}k^{nk^r-r-n}T(D)((k-1)!)^{nk^r}$$
$$= \frac{(k!)^{nk^r}}{nk^{r+n}}T(D).$$

The theorem is thus proved. $\square$

3. Application to interconnected networks

Since the networks studied in [5] are iterated line digraphs of circulants, we state the following definition. The directed circulant graph $C(n, s_1, s_2, \ldots, s_k)$ is defined to be the digraph with n vertices labelled $0, 1, 2, \ldots, n-1$ such that for each vertex i there are k arcs from i to $i+s_1, i+s_2, \ldots, i+s_k \pmod n$, where s_i is an integer and $0 \leqslant s_0 < s_1 < s_2 < \cdots < s_k \leqslant n-1$ (see Fig. 2).

Let us denote the adjacency matrix of $C(n, s_1, s_2, \ldots, s_k)$ by the circulant matrix (see [9, p.16])

$$A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n-1} \\ a_{10} & a_{11} & \cdots & a_{2n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-10} & a_{n-11} & \cdots & a_{n-1n-1} \end{pmatrix},$$

where

$$a_{ij} = \begin{cases} 1, & j-i \in \{s_1, s_2, \ldots, s_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

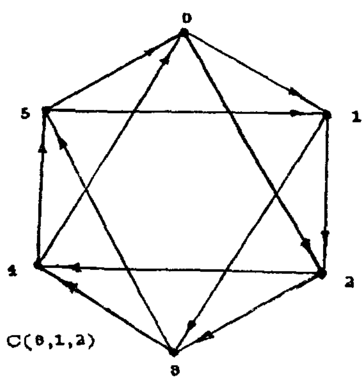

Fig. 2.

Furthermore,

$$P(C(n,s_1,s_2,\dots,s_k),\lambda)=f_\lambda(\varepsilon)f_\lambda(\varepsilon^2)\cdots f_\lambda(\varepsilon^{n-1})f_\lambda(1), \tag{3}$$

where $f_\lambda(x)=\lambda-a_{00}-a_{01}x-\cdots-a_{0n-1}x^{n-1}$.

It is well-known that $C(n,s_1,s_2,\dots,s_k)$ and their line digraph have been widely used in interconnected networks (see [6, 5]), such as de Bruijn digraphs, which is the iterated digraph of $C(n,0,1,2,\dots,n-1)$, and Kautz digraphs, which is the iterated line digraph of $C(n,1,2,\dots,n-1)$. By the explicit expression of $P(C(n,s_1,s_2,\dots,s_k),\lambda)$ and Theorems 1 and 2, we can calculate the number of directed spanning trees and Eulerian tours of $L^r(C(n,s_1,s_2,\dots,s_k))$. For some useful special cases, its explicit expression is very elegant.

Now we give some examples.

Example 1 (*de Bruijn digraphs* (*for the case see Fig.* 1)). Since the adjacency matrix of $C(n,0,1,2,\dots,n-1)$ is J, using the results of [9, p.17), we have

$$P(C(n,0,1,2,\dots,n-1),\lambda)=|\lambda I-J|=\lambda^{n-1}(\lambda-n),$$

where I is the identity matix and J the matrix with each entry equal to 1. By equality (2) and Theorems 1 and 2, we obtain the following results:

$$T(C(n,0,1,2,\dots,n-1))=n^{n-1},$$

$$T(L^r(C(n,0,1,2,\dots,n-1)))=n^{n^{r+1}-n}n^{n-1}=n^{n^{r+1}-1},$$

$$E(L^r(C(n,0,1,2,\dots,n-1)))=(n!)^{n^{r+1}}n^{-(r+2)}.$$

Thus, we reproduce the consequences of [12].

The following results are new.

Example 2 (*Kautz digraphs*). Since the adjacency matrix of $C(n,1,2,\dots,n-1)$ is $J-I$, using the result of [9, p.17], we get

$$P(C(n,1,2,\dots,n-1))=|\lambda I-J+I|=(\lambda-n-1)(\lambda+1)^{n-1}.$$

By equality (2) and Theorems 1 and 2 of this paper, we have

$$T(C(n,1,2,\dots,n-1))=n^{n-1},$$

$$T(L^r(C(n,1,2,\dots,n-1)))=(n-1)^{n(n-1)^r-n}n^{n-1},$$

$$E(L^r(C(n,1,2,\dots,n-1)))=((n-1)!)^{n(n-1)^r}(n-1)^{-n-r}n^{n-2}.$$

Example 3. The circulant $C(n,s_1,s_2)$ is usually called double fixed step networks which is applicable to interconnected networks (see [6]). In general, we can deduce a formulae of $T(C(n,s_1,s_2))$ and $E(C(n,s_1,s_2))$, but the expression of the formulae will not be elegant. Here we prefer to calculate a specific case $T(C(n,1,2))$ (when

$n = 6$, $C(6,1,2)$ appears as G_2^2 in Fig. 4 of [5]):

$$P(C(n,1,2),\lambda) = f_\lambda(\varepsilon)f_\lambda(\varepsilon^2)\cdots f_\lambda(\varepsilon^{n-1})f_\lambda(1),$$

where

$$f_\lambda(\varepsilon) = \lambda - \varepsilon - \varepsilon^2, \quad f_\lambda(1) = \lambda - 2.$$

By equality (2), we have

$$\begin{aligned} T(C(n,1,2)) &= P'(C(n,1,2),\lambda)|_{\lambda=2} \\ &= \prod_{i=1}^{n-1}(2 - \varepsilon^i - \varepsilon^{2i}) \\ &= (-1)^{n-1}\prod_{i=1}^{n-1}(-2 - \varepsilon^i)\prod_{i=1}^{n-1}(1 - \varepsilon^i). \end{aligned}$$

Since $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1) = (x-1)(x-\varepsilon)(x-\varepsilon^2)\cdots(x-\varepsilon^{n-1})$, therefore $x^{n-1} + x^{n-2} + \cdots + x + 1 = (x-\varepsilon)(x-\varepsilon^2)\cdots(x-\varepsilon^{n-1})$. Taking $x = -2$, 1, respectively, in above, we obtain

$$\prod_{i=1}^{n-1}(-2 - \varepsilon^i) = (-1)^{n-1}\frac{1-(-2)^n}{1+2} = \frac{(-1)^{n-1} + 2^n}{3},$$

$$\prod_{i=1}^{n-1}(1 - \varepsilon^i) = n.$$

Thus, we have

$$T(C(n,1,2)) = \frac{(-1)^{n-1} + 2^n}{3}n.$$

By using Theorems 1 and 2, we obtain the number of spanning trees and Eulerian tours of $L^r(C(n,1,2))$ as follows:

$$T(L^r(C(n,1,2))) = 2^{n2^r - n}\frac{(-1)^{n-1} + 2^n}{3}n,$$

$$E(L^r(C(n,1,2))) = 2^{n2^r - r - n}\frac{(-1)^{n-1} + 2^n}{3}.$$

When $n = 6$, the graph $C(6,1,2)$ is considered in [5]. Here we compute the number of spanning trees and Eulerian tours of its iterated line digraphs:

$$\begin{aligned} T(L^r(C(6,1,2))) &= 2^{6\times 2^r - 6} \times \frac{(-1)^{6-1} + 2^6}{3} \times 6 \\ &= 2^{6(2^r-1)} \times 126 \\ &= 64^{(2^r-1)} \times 126, \end{aligned}$$

$$\begin{aligned} E(L^r(C(n,1,2))) &= 2^{6\times 2^r - r - 6} \times \frac{(-1)^{6-1} + 2^6}{3} \\ &= 2^{6\times 2^r - r - 6} \times 21. \end{aligned}$$

Example 4 (*The iterated line digraphs of directed circulants* (*the general case*)). From (2), we have

$$T(C(n,s_1,s_2,\ldots,s_k)) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\prod_{i=1}^{n-1} f_\lambda(\varepsilon^i)|_{\lambda=k}$$
$$= \prod_{i=1}^{n-1} f_k(\varepsilon^i),$$

where $f_\lambda(x)$ is given in (3).

By Theorems 1 and 2, we have

$$T(L^r(C(n,s_1,s_2,\ldots,s_k))) = k^{nk^r-n}\prod_{i=1}^{n-1} f_k(\varepsilon^i),$$

$$E(L^r(C(n,s_1,s_2,\ldots,s_k))) = \frac{1}{n}k^{-r-n}((k)!)^{nk^r}\prod_{i=1}^{n-1} f_k(\varepsilon^i).$$

Finally, we would like to mention that recently in [12] we obtained the explicit expressions of the number of spanning trees and Eulerian tours of generalized de Bruijn graphs. But, in general, the iterated line digraph is not a generalized de Bruijn graph. In fact, the graphs in Examples 2 and 3 are not generalized de Bruijn graphs.

In addition, we can see from the previous examples that the undirected graph G can be changed to a digraph $D(G)$ by replacing each edge by a pair of opposite arcs and they have the same characteristic polynomials, but the line graph of G and the line digraph of $D(G)$ are quite different.

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